From nonlinear to linear elasticity in a coupled rate-dependent/independent system for brittle delamination

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September 19, 2017

Abstract

We revisit the weak, energetic-type existence results obtained in [RT15] for a system for rate-independent, brittle delamination between two visco-elastic, *physically nonlinear* bulk materials and explain how to rigorously extend such results to the case of visco-elastic, *linearly* elastic bulk materials. Our approximation result is essentially based on deducing the MOSCO-convergence of the functionals involved in the energetic formulation of the system. We apply this approximation result in two different situations at small strains: Firstly, to pass from a nonlinearly elastic to a linearly elastic, brittle model on the time-continuous level, and secondly, to pass from a time-discrete to a time-continuous model using an adhesive contact approximation of the brittle model, in combination with a vanishing, super-quadratic regularization of the bulk energy. The latter approach is beneficial if the model also accounts for the evolution of temperature.

2010 Mathematics Subject Classification: 49J53, 49J45, 74H20, 74C05, 74C10, 74M15, 74R10.
Key words and phrases: Adhesive contact, brittle delamination, non-smooth brittle constraint, Kelvin-Voigt visco-elasticity, coupled rate-dependent/rate-independent evolution, semistable energetic solutions.

1 Introduction

In the spirit of generalized standard materials, cf. e.g. [HN75], delamination processes along a prescribed interface Γ_C between two elastic materials $\Omega_+, \Omega_- \subset \mathbb{R}^d$ can be modeled with the aid of an internal delamination variable $z : [0,T] \times \Gamma_C \to [0,1]$, which describes the state of the glue located in Γ_C during a time interval $[0,T]$. In particular, in our notation $z(t,x) = 1$, resp. $z(t,x) = 0$, shall indicate that the glue is fully intact, resp. broken, at $(t,x) \in [0,T] \times \Gamma_C$. Such a type of modeling approach in the framework of delamination dates back to e.g. [Kac88, Fré88]. In the case of a *rate-independent* evolution law for z, analytical results for delamination models have been obtained e.g. in [KMR06, RSZ09] in the case of *adhesive contact* and *brittle delamination* in the framework of the energetic formulation of rate-independent processes. Instead, [RTP15], also in the fully rate-independent setting, constructed for the brittle system *local* (or *semistable energetic*) solutions, i.e. fulfilling a minimality property for the displacements and a semistability inequality for the internal variable, combined with an energy-dissipation inequality, cf. also [Rou13]. The approach in [RTP15] was based on time discretization using an alternate minimization scheme. Semistable energetic solutions to the adhesive contact system were also obtained in [Sca17] by a vanishing-viscosity approach. In [RR11] existence of semistable energetic solutions for an adhesive contact model with rate-independent evolution of the delamination variable was discussed for the first time in combination with other rate-dependent effects: Therein, the displacements are subjected to viscosity and acceleration, and in addition also the evolution of temperature is taken into account. Based on this, [RT15] addressed the existence of (weak, energetic-type) solutions for a *brittle*

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delamination system, extending the isothermal, fully rate-independent model addressed in [RSZ09] to the coupled rate-independent/rate-dependent setting of [RR11]. The aim of this work is to further extend the analytical results that were developed in [RT15] for rate-independent delamination in visco-elastic *physically nonlinear* materials at small strains, to the case of *physically linear* materials at small strains.

More precisely, the existence of solutions to the coupled rate-dependent/independent system for brittle delamination was shown in [RT15] by passing to the limit in an approximate system for *adhesive contact*, under the condition that the elastic energy density $W = W(e)$ fulfilled

$$c|e|^p \le W(e) \le C(|e|^p + 1) \quad \text{with } p > d. \tag{1.1}$$

This kind of nonlinear growth is used in the engineering literature to model strain hardening or softening of so-called power-law materials, see e.g. [Kno77, HK81]. In particular, the exponent $p > d$ is applied at small strains in [Bel84] to describe strain hardening. Yet, for our analytical results in [RT15], the condition $p > d$ also had a very specific, technical motivation. In fact, our analysis relied on the validity of a Hardy inequality, applied to the displacement variable u, which at that time was only available for functions in $W^{1,p}(\Omega;\mathbb{R}^d)$ with $p > d$. In the meantime, an improved version of this Hardy inequality, also valid for $p = 2$, was obtained in [EHDR15], thus making the restriction $p > d$ unnecessary. This was already reflected in [RTP15], where the existence of semistable energetic solutions was shown by a constructive approach combining the adhesive-to-brittle limit and the discrete-to-continuous limit passages in a time discretization scheme. A quadratic growth for the elastic energy density was also allowed in [RT17b], where the existence of solutions to the brittle delamination system in *visco-elastodynamics* (i.e., encompassing inertial effects) was still obtained by passing to the limit in the adhesive contact approximate system.

The aim of this note is to close the gap between the results in [RT15] and those in [RTP15, RT17b]. Namely, we will perform

- **(1)** the limit passage from nonlinear to linear small-strain elasticity in the mechanical force balance for the brittle delamination system;
- **(2)** the joint adhesive-to-brittle, discrete-to-continuous, nonlinearly elastic-to-linearly elastic limit passage in a delamination system at small strains, also encompassing thermal effects.

We do not consider the case of geometrically nonlinear materials, which would be treated in a different way in the framework of finite-strain elasticity, e.g. using tools like polyconvexity.

In Section 2.1, we are going to describe the brittle delamination and adhesive contact systems, confining the discussion to the *quasistatic* (without inertia in the mechanical force balance for the displacements) and isothermal case. Yet, as we discuss in more detail in Section 4.2, it is possible to encompass thermal effects in our analysis, still remaining quasistatic for the displacements. But here, unhampered by the technical problems related to the handling of inertia and temperature, we will focus on the analytical difficulties attached to the adhesive-to-brittle limit. We will then explain the technique for taking the adhesive-to-brittle limit passage in the equation for the displacements first developed in [RT15]. This will help us put into context the main result of this paper, Theorem 2.5, stating the Mosco-convergence of the energy functionals underlying the brittle (small-strain) mechanical force balance from the nonlinearly to linearly elastic case. While Theorem 2.5 will be stated in Section 2.2 and proved throughout Section 3, its applications to the limit passages **(1)** & **(2)** will be carried out in Section 4.

Let us finally fix some notation that will be used throughout the paper: We will denote by $\|\cdot\|_X$ both the norm of a Banach space X and, often, the norm in any power of it, and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. Moreover, we shall often denote by the symbols c, c', C, C' various positive constants, whose meaning may vary from line to line, depending only on known quantities.

2 Our main result: motivation and statement

2.1 The brittle delamination system, its adhesive contact approximation, and the adhesive-to-brittle limit

Let us now gain insight into the PDE system for brittle delamination between two bodies Ω_+ and $\Omega_- \subset \mathbb{R}^d$, $d \geq 2$. We enforce the

$$\text{brittle constraint:} \qquad [\![u(t)]\!] = 0 \quad \text{a.e. on } (0,T) \times \operatorname{supp} z(t) \tag{2.1}$$

where $[\![u]\!] = u^+|_{\Gamma_C} - u^-|_{\Gamma_C}$ is the jump of u across the interface $\Gamma_C = \overline{\Omega_-} \cap \overline{\Omega_+}$, $u^\pm|_{\Gamma_C}$ denoting the traces on Γ_C of the restrictions $u^\pm$ of u to $\Omega_\pm$, and $\operatorname{supp} z$ the support of the delamination variable $z \in L^\infty(\Gamma_C)$, cf. (2.19) ahead. Hence, (2.1) ensures the continuity of the displacements, i.e. $[\![u(t,x)]\!] = 0$, in the (closure of the) set of points where (a portion of) the bonding is still active, i.e. $z(t,x) > 0$, and it allows for displacement jumps only in points $x \in \Gamma_C$ where the bonding is completely broken, where $z(t,x) = 0$. Therefore, (2.1) distinguishes between the crack set $\Gamma_C \setminus \operatorname{supp} z(t)$, where the displacements may jump, and the complementary set with active bonding, where it imposes a transmission condition on the displacements. We also enforce the

$$\text{non-penetration condition:} \qquad [\![u(t)]\!] \cdot \mathbf{n} \geq 0 \quad \text{a.e. on } (0,T) \times \operatorname{supp} z(t) \tag{2.2}$$

with $\mathbf{n}$ the unit normal to Γ_C, oriented from Ω_+ to Ω_-.

The PDE system for brittle delamination between two visco-elastic bodies addressed in this paper consists of the *quasistatic* mechanical force balance for the displacements

$$-\operatorname{div}(\sigma(e, \dot{e})) = F \qquad \text{in } (0,T) \times (\Omega_+ \cup \Omega_-), \tag{2.3a}$$

where $e = e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ is the linearized strain tensor and $\dot{e} = e(\dot{u})$, while F is a time-dependent applied volume force. The stress tensor σ, encompassing the visco-elastic response of the body, is given by the following constitutive law

$$\sigma(e, \dot{e}) = \mathbb{D}\dot{e} + \mathrm{D}W(e),$$

where $\mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ is the symmetric and positive definite viscosity tensor and the elastic energy density $W : \mathbb{R}^{d\times d} \to [0,\infty)$, with Gâteaux derivative $\mathrm{D}W$, is specified by (2.18) below. Equation (2.3a) is supplemented with homogeneous Dirichlet boundary conditions on the Dirichlet part Γ_D of the boundary $\partial\Omega$, where $\Omega := \Omega_+ \cup \Gamma_C \cup \Omega_-$, and subject to an applied traction f on the Neumann part $\Gamma_N = \partial\Omega \setminus \Gamma_D$, i.e.

$$u = 0 \quad \text{on } (0,T) \times \Gamma_D, \qquad \sigma(e,\dot{e})|_{\Gamma_N}\nu = f \quad \text{on } (0,T) \times \Gamma_N, \tag{2.3b}$$

with ν the outward unit normal to $\partial\Omega$. For technical reasons, we will require Γ_D to have positive distance from Γ_C, cf. Assumption 2.3 ahead. The evolution of u and of the delamination parameter z are coupled through the following (formally written) boundary condition on the contact surface Γ_C

$$\sigma(e,\dot{e})|_{\Gamma_C}\mathbf{n} + \partial_u \widetilde{J}_\infty([\![u]\!], z) + \partial I_{C(x)}([\![u]\!]) \ni 0 \quad \text{on } (0,T) \times \Gamma_C, \tag{2.4}$$

where the subdifferential terms render the brittle and non-penetration constraints, respectively. Indeed, $\partial_u \widetilde{J}_\infty : \mathbb{R}^d \times \mathbb{R} \rightrightarrows \mathbb{R}^d$ is the subdifferential (in the sense of convex analysis) of the functional $\widetilde{J}_\infty : \mathbb{R}^d \times \mathbb{R} \to [0,\infty]$ defined by the indicator function of the set individuated by (a slighlty weaker version of) the brittle constraint, namely

$$\widetilde{J}_\infty(v,z) := I_{\{vz=0\}}(v,z) = \begin{cases} 0 & \text{if } vz = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.5}$$

The non-penetration constraint is imposed through the multivalued mapping $C : \Gamma_C \rightrightarrows \mathbb{R}^d$ defined by

$$C(x) := \{v \in \mathbb{R}^d \, : \, v \cdot \mathbf{n}(x) \geq 0\} \qquad \text{for a.a. } x \in \Gamma_C. \tag{2.6}$$

Further coupling is provided by the flow rule for the delamination parameter

$$\partial \mathrm{R}(\dot z) + \partial \mathcal{G}(z) + \partial_z \widetilde{J}_\infty(\llbracket u \rrbracket, z) \ni 0 \quad \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \tag{2.7}$$

featuring the dissipation potential density

$$\mathrm{R}(\dot z) := \begin{cases} a_1|\dot z| & \text{if } \dot z \le 0,\\ \infty & \text{otherwise,}\end{cases}$$

(with $a_1 > 0$ the phenomenological specific energy per area dissipated by disintegrating the adhesive) and $\partial\mathcal{G}$ the (still formally written) subdifferential of a functional $\mathcal{G}$ encompassing a suitable gradient regularization, given in (2.16) below.

The brittle and non-penetration constraints are reflected in the variational formulation of the mechanical force balance for the displacements. To properly give it, we introduce the time-dependent spaces

$$\mathbf{V}_z^q(t) := \{ v \in W^{1,q}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \,:\, \llbracket v \rrbracket = 0 \text{ a.e. on } \operatorname{supp} z(t) \subset \Gamma_{\mathrm{C}} \text{ and } \llbracket v(x) \rrbracket \in C(x) \text{ for a.a. } x\in\Gamma_{\mathrm{C}} \},$$

where the exponent $q > 1$ depends on the growth properties of the density W and we use the notation $W^{1,q}_{\mathrm{D}}(A;\mathbb{R}^d)$ for the space of $W^{1,q}$-functions on a domain A with null trace on Γ_{D}. In this work, we will in particular deal with the cases $q = p > d$ and $q = 2$. Thus, the weak formulation of (2.3) reads

$$\begin{aligned}
&u(t) \in \mathbf{V}_z^q(t) \quad \text{for a.a. } t\in(0,T),\\
&\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{D}e(\dot u(t)) + \mathrm{D}W(e(u(t)))\big) : e(v-u(t))\,\mathrm{d}x \ge \langle L(t), v-u(t)\rangle \text{ for all } v\in\mathbf{V}_z^q(t), \text{ for a.a. } t\in(0,T),
\end{aligned} \tag{2.8}$$

with $L : (0,T) \to W^{1,q}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*$ a functional subsuming the external forces F and f, i.e.

$$\langle L(t), v\rangle := \int_\Omega F(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} f(t)\cdot v\,\mathrm{d}S\,; \tag{2.9}$$

more details on the above duality pairing and the conditions on the forces F and f will be given in Sec. 4.1. In this paper, along the footsteps of [Rou09, RT17a], we will weakly formulate the coupled rate-dependent/independent system (2.3, 2.4, 2.7) by means of an extension of the concept of semistable energetic solution from [Rou13]. As we will see in Definition 4.1 ahead, the *semistable energetic* solutions of system (2.3, 2.4, 2.7) are defined by fulfilling the weak mechanical force balance for the displacements (2.8) combined with a suitable energy-dissipation inequality and a semistability condition, weakly rendering the flow rule (2.7).

In [RT15] we showed the existence of semistable energetic solutions of the brittle system, by passing to the limit in an approximate system where the brittle constraint (2.1) is penalized by the

$$\text{adhesive contact term:} \qquad \int_{\Gamma_{\mathrm{C}}} J_k(\llbracket u \rrbracket, z)\,\mathrm{d}\mathcal{H}^{d-1}(x) \text{ with } J_k(\llbracket u \rrbracket, z) := \frac{k}{2} z|\llbracket u \rrbracket|^2 \quad \text{for } k>0, \tag{2.10}$$

featured in the energy functional underlying the mechanical force balance for the displacements. Above, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. In fact, the existence of *energetic solutions* to the purely rate-independent brittle system was proved in [RSZ09] by passing to the limit in this adhesive contact approximation, as the parameter $k \to \infty$. For our coupled rate-dependent/independent brittle system, the adhesive contact approximation consists of the mechanical force balance (2.3) for the displacements coupled with the following contact surface condition and flow rule for the delamination parameter

$$\sigma(e,\dot e)|_{\Gamma_{\mathrm{C}}}\mathbf{n} + \partial_u J_k(\llbracket u \rrbracket, z) + \partial I_{C(x)}(\llbracket u \rrbracket) \ni 0 \quad \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \tag{2.11}$$

$$\partial\mathrm{R}(\dot z) + \partial\mathcal{G}(z) + \partial_z J_k(\llbracket u \rrbracket, z) \ni 0 \quad \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \tag{2.12}$$

which replace (2.4) and (2.7), respectively. Accordingly, the weak formulation of the mechanical force balance for the adhesive contact system (2.3, 2.11, 2.12) reads

$$\int_{\Omega\backslash\Gamma_{\mathrm{C}}}\!\!\big(\mathbb{D}e(\dot u(t))+\mathrm{D}W(e(u(t)))\big):e(v-u(t))\,\mathrm{d}x+\int_{\Gamma_{\mathrm{C}}}kz(t)\,[\![u(t)]\!]\,[\![v-u(t)]\!]\,\mathrm{d}\mathcal{H}^{d-1}(x)\geq\ \langle L(t),v-u(t)\rangle$$
$$\text{for all } v\in\mathbf{V}^q,\ \text{for a.a. } t\in(0,T), \tag{2.13}$$

where the (no longer time-dependent) space for the test functions now only encompasses the non-penetration condition (2.2), i.e.

$$\mathbf{V}^q:=\{v\in W^{1,q}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)\ :\ [\![v(x)]\!]\in C(x)\text{ for a.a. } x\in\Gamma_{\mathrm{C}}\}\,.$$

The limit-passage argument for the adhesive-to-brittle limit developed in [RSZ09] was based on the Evolutionary Gamma-convergence theory for (purely) rate-independent systems from [MRS08]: Basically, it only necessitated the Gamma-convergence of the underlying energy and dissipation functionals, combined with a mutual recovery sequence condition that ensured the limit passage in the global stability condition. For coupled rate-dependent/independent systems, it is not sufficient to solely rely on the abstract toolbox of [MRS08]: In particular, in our specific context, the Gamma-convergence of the energies no longer guarantees the limit passage, as $k\to\infty$, from the weak mechanical force balance for the displacements (2.13) to its brittle analogue (2.8). For that, given a sequence of semistable energetic solutions $(u_k,z_k)_k$ converging to a pair (u,z), which is a candidate semistable energetic solution of the brittle system, it is indeed necessary to construct, for every admissible test function $v\in\mathbf{V}^q_z(t)$ for the brittle mechanical force balance (2.8), with $t\in(0,T)$ fixed, a sequence $(v_k)_k$ of test functions for (2.13) such that

1. $(v_k)_k$ converge to v in a suitable sense, ensuring the limit passage in the bulk terms of (2.13);
2. the functions v_k also satisfy the non-penetration condition (2.2);
3. there holds
$$\limsup_{k\to\infty}\int_{\Gamma_{\mathrm{C}}}kz_k(t)\,[\![u_k(t)]\!]\,[\![v_k-u_k(t)]\!]\,\mathrm{d}\mathcal{H}^{d-1}(x)\leq 0\,.$$

Since $\liminf_{k\to\infty}\int_{\Gamma_{\mathrm{C}}}kz_k(t)|[\![u_k(t)]\!]|^2\,\mathrm{d}\mathcal{H}^{d-1}(x)\geq 0$ for almost all $t\in(0,T)$, it is immediate to check that the above property is ensured as soon as

$$\limsup_{k\to\infty}\int_{\Gamma_{\mathrm{C}}}kz_k(t)\,[\![u_k(t)]\!]\,[\![v_k]\!]\,\mathrm{d}\mathcal{H}^{d-1}(x)\leq 0\,. \tag{2.14}$$

In [RT15] we were able to construct a sequence $(v_k)_k$ complying with (2.14), starting from a test function v such that $[\![v]\!]=0$ a.e. on $\operatorname{supp}z(t)$, by modifying v in such a way that the support of the obtained $[\![v_k]\!]$ fitted to the null set of z_k, approximating z. This construction hinged on two crucial ingredients:

1. First, we preliminarily obtained refined convergence properties of the delamination variables $(z_k)_k$. In particular, we proved the *support convergence*
$$\operatorname{supp}z_k(t)\subset\operatorname{supp}z(t)+B_{\rho_k}(0)\quad\text{and}\quad\rho_k\to0\text{ as }k\to\infty, \tag{2.15}$$
at every $t\in(0,T)$ via arguments from geometric measure theory. In fact, our proof of (2.15) heavily relied on the following, specific choice for the gradient regularizing term for the delamination flow rule
$$\mathcal{G}(z):=\begin{cases}\mathrm{b}|\mathrm{D}z|(\Gamma_{\mathrm{C}}) & \text{if } z\in\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}),\\ \infty & \text{otherwise,}\end{cases} \tag{2.16}$$
with $\mathrm{b}>0$, $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ the set of the special bounded variation functions on Γ_{C}, taking values in $\{0,1\}$, and $|\mathrm{D}z|(\Gamma_{\mathrm{C}})$ the variation on Γ_{C} of the Radon measure $\mathrm{D}z$. The set $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ thus only consists of characteristic functions of sets with finite perimeter in Γ_{C}, and the total variation $|\mathrm{D}z|(\Gamma_{\mathrm{C}})$ of $z=\chi_Z\in\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ is given by the perimeter of Z in Γ_{C}. With (2.16) we thus imposed that z only takes the values 0 and 1, i.e. we encompassed in the model only two states of the bonding between Ω_+ and Ω_-, the fully effective and the completely ineffective ones. Relying

on the information $z_k \in \{0,1\}$ and on the support convergence (2.15), we in fact constructed a sequence $(v_k)_k$ such that

$$z_k(t)|[\![v_k(t)]\!]|^2 = 0 \quad \text{for all } k \in \mathbb{N} \text{ and all } t \in [0,T]. \tag{2.17}$$

2. Second, for establishing the convergence properties of the recovery sequence of test functions for the displacements, we resorted to a Hardy inequality given in [Lew88] for closed sets of arbitrarily low regularity, but applicable only to functions in $W^{1,p}(\Omega;\mathbb{R}^d)$, with $p > d$. To enforce this integrability property for the gradients of the displacements, we thus had to impose the growth condition (1.1) on the elastic energy density and, accordingly, x to consider the variational formulation of the adhesive contact and of the brittle equations for the displacements in the spaces $\mathbf{V}^p$ and $\mathbf{V}^p_z(t)$, respectively. However, this condition can be weakened to *quadratic* growth in view of the improved Hardy's inequality recently proved in [EHDR15].

As a matter of fact, our construction of recovery test functions did guarantee the Mosco-convergence of the energy functionals underlying the adhesive contact mechanical force balance (2.13) to that of the brittle mechanical force balance (2.8).

Indeed, in Sec. 2.2, we are going to state the main result of this paper in terms of Mosco-convergence of functionals. This result will ensure the passage from elastic energy densities with $(p{>}d)$-growth to quadratic densities in the following two situations:

1. in the brittle delamination system: for this, we will resort to the convergence of the functionals $(\Phi_k)_k$ to Φ_∞, cf. (2.21) & (2.23);
2. jointly with the adhesive-to-brittle and discrete-to-continuous limit passage in thermo-visco-elastic delamination systems: for this, we will resort to the convergence of the functionals $(\Phi_k^{\mathrm{adh}})_k$ to Φ_∞, cf. (2.22) & (2.23).

2.2 Our main result

Definition of Mosco-convergence. We recall the definition from, e.g. [Att84, Sec. 3.3, p. 295]): Given a Banach space X and proper functionals Φ_k, $\Phi_\infty : \mathbb{R} \to (-\infty,\infty]$, $k \in \mathbb{N}$, we say that the sequence $(\Phi_k)_k$ Mosco-converges to Φ as $k \to \infty$ if the following conditions hold:

- lim inf-inequality: for every $u \in X$ and all $(u_k)_k \subset X$ there holds

$$u_k \rightharpoonup u \text{ weakly in } X \;\Rightarrow\; \liminf_{k\to\infty} \Phi_k(u_k) \geq \Phi_\infty(u);$$

- lim sup-inequality: for every $v \in X$ there exists a sequence $(v_k)_k \subset X$ such that

$$v_k \to v \text{ strongly in } X \quad \text{and} \quad \limsup_{k\to\infty} \Phi_k(v_k) \leq \Phi_\infty(v).$$

The functionals. Throughout the paper, we will consider elastic energy densities of the type

$$\begin{aligned} &W_q : \mathbb{R}^{d\times d} \to [0,\infty) \text{ convex, differentiable, and such that} \\ &\exists\, c_q,\, C_q > 0\ \forall\, e \in \mathbb{R}^{d\times d} : \quad c_q|e|^q \leq W_q(e) \leq C_q(|e|^q{+}1) \end{aligned} \tag{2.18}$$

for some $q \in (1,\infty)$ and the associated integral functionals on $\Omega\backslash\Gamma_{\mathrm{C}}$. We will also consider the integral functional induced by J_k from (2.10), i.e.

$$\mathcal{J}_k(v,z) := \int_{\Gamma_{\mathrm{C}}} J_k(v(x), z(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)\,,$$

whose domain of definition depends on the choice of q from (2.18), cf. Remark 2.1 for more details. While $\mathcal{J}_k$ will contribute to Φ_k^{adh}, the functionals Φ_k and Φ_∞ will feature a term $\mathcal{J}_\infty$ accounting for the brittle constraint (2.1), which in turn involves the closed set $\operatorname{supp} z$. We will consider $\mathcal{J}_\infty$ to be defined for $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, which can be thus identified with the characteristic function of a finite perimeter set Z. In a measure-theoretic sense, $\operatorname{supp} z$ is given by

$$\operatorname{supp} z := \bigcap\{A \subset \Gamma_{\mathrm{C}} \subset \mathbb{R}^{d-1};\ A \text{ closed },\ \mathcal{H}^{d-1}(Z\backslash A) = 0\}. \tag{2.19}$$

We now define

$$\mathcal{J}_\infty : L^1(\Gamma_{\rm C};\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_{\rm C};\{0,1\}) \to [0,\infty] \qquad \mathcal{J}_\infty(v,z) := \begin{cases} 0 & \text{if } v=0 \ \mathcal{H}^{d-1}\text{-a.e. on } \operatorname{supp} z, \\ \infty & \text{otherwise.} \end{cases} \tag{2.20}$$

Finally, we introduce the integral functional induced by the indicator functions of the sets $C(x)$ from (2.6), i.e.

$$\mathcal{I}_C : L^1(\Gamma_{\rm C};\mathbb{R}^d) \to [0,\infty] \qquad \mathcal{I}_C(v) := \int_{\Gamma_{\rm C}} I_{C(x)}(v(x))\,\mathrm{d}\mathcal{H}^{d-1}(x)\,.$$

Then, we define the functionals

$$\Phi_k : H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_{\rm C};\{0,1\}) \to [0,\infty] \quad \text{given by}$$
$$\Phi_k(u,z) := \begin{cases} \int_{\Omega\backslash\Gamma_{\rm C}} \big(W_2(e(u)) + \frac{1}{k^p} W_p(e(u))\big)\,\mathrm{d}x + \mathcal{J}_\infty([\![u]\!],z) & \text{if } u \in W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \quad \text{with } p>d, \tag{2.21}$$

$$\Phi_k^{\rm adh} : H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times L^1(\Gamma_{\rm C}) \to [0,\infty] \quad \text{given by}$$
$$\Phi_k^{\rm adh}(u,z) := \begin{cases} \int_{\Omega\backslash\Gamma_{\rm C}} \big(W_2(e(u)) + \frac{1}{k^p} W_p(e(u))\big)\,\mathrm{d}x + \mathcal{J}_k([\![u]\!],z) & \text{if } u \in W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \quad \text{with } p>d. \tag{2.22}$$

We will show that, given a sequence $(z_k)_k \subset \mathrm{SBV}(\Gamma_{\rm C};\{0,1\})$ and suitably converging to some $z \in \mathrm{SBV}(\Gamma_{\rm C};\{0,1\})$ (cf. Theorem 2.5 below), both functionals $\Phi_k(\cdot,z_k)$ and $\Phi_k^{\rm adh}(\cdot,z_k)$ MOSCO-converge in the $H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$-topology, as $k\to\infty$, to the functional $\Phi_\infty(\cdot,z)$ defined by

$$\Phi_\infty : H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_{\rm C};\{0,1\}) \to [0,\infty], \quad \Phi_\infty(u,z) := \int_{\Omega\backslash\Gamma_{\rm C}} W_2(e(u))\,\mathrm{d}x + \mathcal{J}_\infty([\![u]\!],z)\,. \tag{2.23}$$

Remark 2.1. 1. Due to the condition $p>d$ and to trace theorems, for every $u \in W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ there holds

$$[\![u]\!] \in W^{1-1/p,p}(\Gamma_{\rm C};\mathbb{R}^d) \subset \mathrm{C}^0(\Gamma_{\rm C})\,. \tag{2.24}$$

Therefore, for the term $\mathcal{J}_k([\![u]\!],z)$ to be well defined, it is in principle sufficient to have $z\in L^1(\Gamma_{\rm C})$.

2. As already mentioned, in [RT15] we performed the adhesive-to-brittle limit passage in the mechanical force balance staying in the context of nonlinear (small-strain) elasticity, with an elastic energy having p-growth, with $p>d$. In fact, we proved the MOSCO-convergence of the functionals (w.r.t. the variable u, with the second entry given by a sequence $(z_k)_k$ in $\mathrm{SBV}(\Gamma_{\rm C};\{0,1\})$ suitably converging to some z)

$$\Phi_k^{{\rm adh},p} : W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times L^1(\Gamma_{\rm C}) \to [0,\infty) \;\; \Phi_k^{{\rm adh},p}(u,z) := \int_{\Omega\backslash\Gamma_{\rm C}} W_p(e(u))\,\mathrm{d}x + \mathcal{J}_k([\![u]\!],z),$$

to the functional

$$\widetilde{\Phi}^p_\infty : W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times L^1(\Gamma_{\rm C}) \to [0,\infty] \;\; \widetilde{\Phi}^p_\infty(u,z) := \int_{\Omega\backslash\Gamma_{\rm C}} W_p(e(u))\,\mathrm{d}x + \widetilde{\mathcal{J}}_\infty([\![u]\!],z),$$

with $\widetilde{\mathcal{J}}_\infty$ the integral functional induced by the indicator function $\widetilde{J}_\infty$ from (2.5). Observe that, in view of (2.24), for $u \in W^{1,p}_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ there holds

$$z[\![u]\!] = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\rm C} \iff [\![u]\!] = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \operatorname{supp} z, \quad \text{hence } \widetilde{\mathcal{J}}_\infty([\![u]\!],z) = \mathcal{J}_\infty([\![u]\!],z)\,.$$

Instead, for the functional Φ_∞, defined with $u \in H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ it is essential to have the contribution with $\mathcal{J}_\infty$, which enforces constraint (2.1) in terms of $\operatorname{supp} z$, stronger than $z[\![u]\!]=0$ a.e. on $\Gamma_{\rm C}$. In fact, our argument for MOSCO-convergence relies on the support convergence (2.15).

Assumptions. Let us now specify our geometric assumptions on the domain Ω, as well as the properties required of a sequence $(z_k)_k \subset \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, converging to some $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, to ensure that the functionals $\Phi_k(\cdot,z_k)$ and $\Phi_k^{\mathrm{adh}}(\cdot,z_k)$ MOSCO-converge to $\Phi_\infty(\cdot,z)$. In order to obtain a result as independent as possible from the problem of passing to the limit in the *coupled* system for brittle delamination, we will directly impose here certain additional regularity properties on $(z_k)_k$ and z, which are in fact induced by semistability, see Sec. 4.1.

We will suppose that the Dirichlet boundary Γ_{D} and the finite perimeter sets Z_k and Z associated with z_k and z enjoy a regularity property, which prevents outward cusps, introduced by Campanato as the Property $\mathfrak{a}$, cf. e.g. [Cam63, Cam64], and also known as *lower density estimate* in e.g. [FF95, AFP05]. We recall it in the following definition.

Definition 2.2 (Property $\mathfrak{a}$). *A set $M \subset \mathbb{R}^n$ has the Property $\mathfrak{a}$ if there exists a constant C such that*

$$\forall\, y \in M \;\; \forall\, \rho_\star > 0: \quad \mathcal{L}^n(M \cap B_{\rho_\star}(y)) \geq C\rho_\star^n\,. \tag{2.25}$$

Here, $B_{\rho_\star}(y)$ denotes the open ball of radius $\rho_\star$ with center in y.

We now fix our conditions on the domain Ω.

Assumption 2.3. *We suppose that*

$$\Omega \subset \mathbb{R}^d,\ d \geq 2, \text{ is bounded, } \Omega_-,\ \Omega_+,\ \Omega \text{ are Lipschitz domains, } \Omega_+ \cap \Omega_- = \emptyset\,, \tag{2.26a}$$

$$\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}},\ \text{ s.th. } \Gamma_{\mathrm{N}} = \partial\Omega\backslash\Gamma_{\mathrm{D}},\ \Gamma_{\mathrm{D}} \subset \partial\Omega \text{ is closed with Property } \mathfrak{a}, \text{ and} \tag{2.26b}$$

$$\Gamma_{\mathrm{D}} \cap \overline{\Gamma}_{\mathrm{C}} = \emptyset,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_-) > 0\,,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_+) > 0\,,\ \mathrm{dist}(\Gamma_{\mathrm{D}},\Gamma_{\mathrm{C}}) = \gamma > 0\,, \tag{2.26c}$$

$$\begin{aligned}\Gamma_{\mathrm{C}} = \overline{\Omega}_- \cap \overline{\Omega}_+ \subset \mathbb{R}^{d-1} \text{ is a ``flat'' surface, i.e. contained in a hyperplane of } \mathbb{R}^d,\\ \text{such that, in particular, } \mathcal{H}^{d-1}(\Gamma_{\mathrm{C}}) = \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}) > 0\,,\end{aligned} \tag{2.26d}$$

where $\mathcal{H}^{d-1}$, resp. $\mathcal{L}^{d-1}$, denotes the $(d-1)$-dimensional Hausdorff measure, resp. Lebesgue measure.

Here, the condition that Γ_{C} is contained in a hyperplane has no substantial role in our analysis, but to simplify arguments and notation.

As for the functions $(z_k)_k,\ z \subset \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, in addition to weak* convergence in $\mathrm{SBV}(\Gamma_{\mathrm{C}})$ we will suppose that they fulfill a *lower density estimate*, holding *uniformly* w.r.t. parameter $k \in \mathbb{N} \cup \{\infty\}$.

Assumption 2.4. *There are constants $R, \mathfrak{a}(\Gamma_{\mathrm{C}}) > 0$ such that for every $k \in \mathbb{N} \cup \{\infty\}$ there holds*

$$\forall\, y \in \operatorname{supp} z_k \;\; \forall\, \rho_\star > 0: \quad \mathcal{L}^{d-1}(Z_k \cap B_{\rho_\star}(y)) \geq \begin{cases} \mathfrak{a}(\Gamma_{\mathrm{C}})\rho_\star^{d-1} & \text{if } \rho_\star < R, \\ \mathfrak{a}(\Gamma_{\mathrm{C}})R^{d-1} & \text{if } \rho_\star \geq R, \end{cases} \tag{2.27}$$

where Z_k is the finite perimeter set such that $z_k = \chi_{Z_k}$.

As we will see in Sec. 3.1, this condition, combined with the weak* convergence in $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, ensures the support convergence (2.15) for the functions z_k.

We are now in a position to state the main result of this paper.

Theorem 2.5. *Under Assumption 2.3, let $(z_k)_k, z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ fulfill as $k \to \infty$*

$$z_k \overset{*}{\rightharpoonup} z \text{ in } \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \tag{2.28}$$

and Assumption 2.4. Then, the functionals $\Phi_k(\cdot,z_k)$ and $\Phi_k^{\mathrm{adh}}(\cdot,z_k)$ MOSCO-converge as $k \to \infty$ to $\Phi_\infty(\cdot,z)$, with respect to the topology of $H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$.

Its proof, carried out in Section 3, is based on a nontrivial adaptation of the arguments for the aforementioned MOSCO-convergence result from [RT15].

3 Proof of Theorem 2.5

Let $(z_k)_k$, $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ fulfill the conditions of Theorem 2.5. In order to prove Mosco-convergence of the functionals $\Phi_k(\cdot,z_k)$ and $\Phi_k^{\mathrm{adh}}(\cdot,z_k)$ to $\Phi_\infty(\cdot,z)$, we have to check the lim inf- and the lim sup-estimates. While the proof of the latter is more involved and will be carried out throughout Sections 3.1 and 3.2, the argument for the former will be developed in the following lines. It relies on this key result.

Lemma 3.1 ([RT17b], Lemma 4.5)**.** *Let $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ and let $Z \subset \Gamma_{\mathrm{C}}$ be the associated finite perimeter set such that $z = \chi_Z$. Suppose that z fulfills the lower density estimate* (2.25)*. Then,*

$$\mathcal{H}^{d-1}(\operatorname{supp} z\backslash Z) = 0. \tag{3.1}$$

The $\liminf$-estimate. Let (u_k), $u \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ fulfill $u_k \rightharpoonup u$. Since W_2 is convex and continuous on $H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ and since $W_p \geq 0$ by (2.18), we have

$$\liminf_{k\to\infty} \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \Big(W_2(e(u_k)) + \frac{1}{k^p} W_p(e(u_k)) \Big)\,\mathrm{d}x \geq \int_{\Omega\backslash\Gamma_{\mathrm{C}}} W_2(e(u))\,\mathrm{d}x\,.$$

We now distinguish the analysis for $\Phi_k(\cdot,z_k)$ from that for $\Phi_k^{\mathrm{adh}}(\cdot,z_k)$, cf. (2.21) & (2.22).

(i) We may of course suppose that $\sup_k \Phi_k(u_k,z_k) \leq C < \infty$. Therefore, we have

$$\sup_{k\in\mathbb{N}} \mathcal{J}_\infty(\llbracket u_k \rrbracket, z_k) \leq C \quad \text{hence} \quad \llbracket u_k \rrbracket = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \operatorname{supp} z_k\,.$$

Since $z_k \to z$ in $L^q(\Gamma_{\mathrm{C}})$ for every $1 \leq q < \infty$ by (2.28), and since $\llbracket u_k \rrbracket \to \llbracket u \rrbracket$ in $L^2(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ by the compact embedding $H^1(\Omega;\mathbb{R}^d) \subset L^2(\Gamma_{\mathrm{C}};\mathbb{R}^d)$, we find a subsequence $(z_k, \llbracket u_k \rrbracket)_k$ converging pointwise a.e. in Γ_{C} to $(z, \llbracket u \rrbracket)$. More precisely, along this subsequence it holds $0 = z_k\llbracket u_k \rrbracket \to z\llbracket u \rrbracket$ a.e. in Γ_{C} and hence we conclude

$$z\llbracket u \rrbracket = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{C}}, \text{ which implies } \llbracket u \rrbracket = 0 \ \mathcal{H}^{d-1}\text{-a.e. on } \operatorname{supp} z \tag{3.2}$$

thanks to (3.1). Therefore,

$$\liminf_{k\to\infty} \mathcal{J}_\infty(\llbracket u_k \rrbracket, z_k) \geq 0 = \mathcal{J}_\infty(\llbracket u \rrbracket, z)\,,$$

which concludes the proof of the lower semicontinuity estimate.

(ii) From $\sup_k \Phi_k^{\mathrm{adh}}(u_k,z_k) \leq C < \infty$ we now infer that $\sup_{k\in\mathbb{N}} \mathcal{J}_k(\llbracket u_k \rrbracket, z_k) \leq C$, which again yields (3.2), because of $0 \leq \int_{\Gamma_{\mathrm{C}}} z_k |\llbracket u_k \rrbracket|^2 \,\mathrm{d}\mathcal{H}^{d-1}(x) \leq C/k \to 0$. Then, also

$$\liminf_{k\to\infty} \mathcal{J}_k(\llbracket u_k \rrbracket, z_k) \geq 0 = \mathcal{J}_\infty(\llbracket u \rrbracket, z)\,.$$

Outline of the proof of the $\limsup$-estimate. Let $v \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ fulfill $\Phi_\infty(v,z) < \infty$: in particular, z and v satisfy the brittle constraint (2.1). It is our task to construct a sequence $(v_k)_k$ with the following properties:

$$\begin{aligned} &v_k \in W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \text{ for all } k\in\mathbb{N},\ \sup_k \Phi_k(v_k,z_k) < \infty, \ \text{ and} \\ &v_k \to v \text{ in } H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \ \&\ \Phi_k(v_k,z_k) \to \Phi_\infty(v,z) \qquad \text{as } k\to\infty. \end{aligned} \tag{3.3}$$

Obviously, in order to improve the regularity of $v \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ to $W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d)$ with $p > d$, v has to be mollified. For this, we will introduce a mollification operator $M^\pm_{\varepsilon_k}$, with a vanishing sequence $(\varepsilon_k)_k$, which involves the H^1-extension of $v|_{\Omega_\pm}$ from $\Omega_\pm$ to $\mathbb{R}^d$ and the convolution with a mollifier $\eta_{\varepsilon_k} \in C_0^\infty(\mathbb{R}^d)$. However, in general, the convolution of $v|_{\Omega_\pm}$ with a mollifier $\eta_k \in C_0^\infty(\mathbb{R}^d)$ will spoil its zero-trace on the Dirichlet boundary $\Gamma_{\mathrm{D}} \cap \overline{\Omega_\pm}$. In order to construct an element of $W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d)$ one has to set $v|_{\Omega_\pm}$ to zero in a sufficiently large, k-dependent neighborhood $\Gamma_{\mathrm{D}} + B_{r_k}(0)$ of Γ_{D}, before convolving with η_k. For this modification of a function $v \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d)$, leading to a function with zero values in a neighborhood of radius ρ of a closed set $M \subset \overline{\Omega}$, we will apply a suitably defined recovery operator that is a function of the radius ρ, of the points in M, and of the elements in $H^1_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d)$. Namely,

$$\mathfrak{Rec} : \{\rho \in [0,\infty)\} \times M \times H^1_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d) \to \{\tilde{v} \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma;\mathbb{R}^d) \ : \ \operatorname{supp} \tilde{v} \subset \Omega\backslash(M + B_\rho(0))\};$$

its definition is given in Def. 3.3 below. The now suitably mollified function $\tilde{v}_k$ given by $\tilde{v}_k|_{\Omega_\pm} = \tilde{v}_k^\pm := \eta_k * \mathfrak{Rec}(r_k, \Gamma_{\mathrm{D}} \cap \overline{\Omega_\pm}, v|_{\Omega_\pm}) \in W^{1,p}_{\mathrm{D}}(\Omega_\pm;\mathbb{R}^d)$, with a vanishing sequence $(r_k)_k$, will have to be further modified in such a way that the brittle constraint (2.1) is satisfied with the given sequence $(z_k)_k$. For this, the recovery operator $\mathfrak{Rec}$ will be once more applied to the triple $(\rho_k, \operatorname{supp} z, \tilde{v}_k^{\mathrm{anti}})$, where $\tilde{v}_k^{\mathrm{anti}}$ is the antisymmetric part of $\tilde{v}_k$, cf. (3.19), and

$$\rho_k := \inf\{\rho \in [0,\infty), \ \operatorname{supp} z_k \subset \operatorname{supp} z + B_{\rho_k}(0)\}\,. \tag{3.4}$$

In other words, the construction of the recovery sequence $(v_k)_k$ complying with (3.3) consists of the following three steps:

Step 1: Set $v|_{\Omega_\pm}$ to zero in $\Gamma_{\mathrm{D}} + B_{r_k}(0)$ using $\mathfrak{Rec}$, with a vanishing sequence $(r_k)_k$: this yields

$$\mathfrak{Rec}(r_k, \Gamma_{\mathrm{D}}^\pm, v|_{\Omega_\pm}), \quad \text{where } \Gamma_{\mathrm{D}}^\pm := \Gamma_{\mathrm{D}} \cap \overline{\Omega_\pm}. \tag{3.5a}$$

Here, the vanishing sequence $(r_k)_k$ has to be chosen in such a way that $(\Gamma_{\mathrm{D}} + B_{r_k}(0)) \cap \Gamma_{\mathrm{C}} = \emptyset$. This is possible thanks to Assumption (2.26c), which provides that $\operatorname{dist}(\Gamma_{\mathrm{D}}, \Gamma_{\mathrm{C}}) = \gamma > 0$.

Step 2: Mollify $\mathfrak{Rec}(r_k, \Gamma_{\mathrm{D}}^\pm, v|_{\Omega_\pm})$ using a suitably defined mollification operator $M_{\varepsilon_k}^\pm \in C_0^\infty(\mathbb{R}^d)$ for a vanishing sequence $(\varepsilon_k)_k$: this results in

$$\tilde{v}_k \in W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \ \text{ with } \ \tilde{v}_k^\pm := M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\mathrm{D}}^\pm, v|_{\Omega_\pm})). \tag{3.5b}$$

Step 3: Adapt $\tilde{v}_k$ to z_k in such a way as to obtain a sequence $(v_k)_k$ satisfying

$$z_k [\![v_k]\!] = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{C}} \ \text{ for each } k \in \mathbb{N}. \tag{3.6}$$

The technical tools for this construction will be provided in Section 3.1, whereas in Section 3.2 we will carry out the proof that the sequence $(v_k)_k$ indeed converges to v as stated in (3.3), cf. Theorem 3.7.

3.1 Preliminary definitions and results

We start by introducing the mollification operators. Since $\Omega_\pm \subset \mathbb{R}^d$ are Lipschitz domains, by [Ada75, p. 91, Thm. 4.32], they are extension domains (for Sobolev functions); we introduce the linear extension operator

$$\begin{aligned}
&E_\pm : H^1(\Omega_\pm;\mathbb{R}^d) \to H^1(\mathbb{R}^d;\mathbb{R}^d) \quad \text{with the properties:}\\
&\bullet\ \forall\, v \in H^1(\Omega_\pm;\mathbb{R}^d): \quad E_\pm(v)(x) = v(x) \text{ a.e. in } \Omega_\pm,\\
&\bullet\ \exists C_\pm > 0\ \forall\, v \in H^1(\Omega_\pm;\mathbb{R}^d): \quad \|E_\pm(v)\|_{H^1(\mathbb{R}^d;\mathbb{R}^d)} \le C_\pm \|v\|_{H^1(\Omega_\pm;\mathbb{R}^d)}\,.
\end{aligned} \tag{3.7}$$

In order to define a suitable mollification operator, we make use of the standard mollifier $\eta_1 \in C_0^\infty(\mathbb{R}^d)$, cf. e.g. [Ada75, p. 29, 2.17],

$$\eta_1(x) := \begin{cases} \zeta \exp\big(-1/(1-|x|^2)\big) & \text{if } |x| < 1,\\ 0 & \text{if } |x| \ge 1, \end{cases} \tag{3.8a}$$

with a constant $\zeta > 0$ such that $\int_{\mathbb{R}^d} \eta_1(x)\,\mathrm{d}x = 1$, and for $\varepsilon > 0$ we set

$$\eta_\varepsilon(x) := \varepsilon^{-d}\eta_1(x/\varepsilon)\,. \tag{3.8b}$$

The mollification operator $M_\varepsilon^\pm$: Now, for $\varepsilon > 0$ we define the mollification operator

$$M_\varepsilon^\pm : H^1(\Omega_\pm;\mathbb{R}^d) \to C^\infty(\Omega_\pm;\mathbb{R}^d), \quad M_\varepsilon^\pm(v) := \eta_\varepsilon * E_\pm(v)|_{\Omega_\pm} = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)E_\pm(v)(y)\,\mathrm{d}y\Big|_{\Omega_\pm} \tag{3.9}$$

and collect its properties in the following result.

Proposition 3.2 (Properties of $M_\varepsilon^\pm$)**.** *Let $p \in (1,\infty)$ fixed.*

1. *For every $\varepsilon > 0$ the linear operator $M_\varepsilon^\pm : H^1(\Omega_\pm;\mathbb{R}^d) \to H^1(\Omega_\pm;\mathbb{R}^d)$ satisfies*
$$\exists \overline{C} > 0\ \forall v \in H^1(\Omega_\pm;\mathbb{R}^d) : \qquad M_\varepsilon^\pm(v)\|_{H^1(\Omega_\pm;\mathbb{R}^d)} \le \overline{C}\|v\|_{H^1(\Omega_\pm;\mathbb{R}^d)}\,. \tag{3.10}$$
2. *Consider a sequence $\varepsilon \to 0$ and let $v \in H^1(\Omega_\pm;\mathbb{R}^d)$. Then, $M_\varepsilon^\pm v \to v$ in $H^1(\Omega_\pm;\mathbb{R}^d)$.*
3. *Let $p > d$ fixed. There is a constant $C_{d,p} > 0$, only depending on Ω, on d, and p, such that for all $v \in H^1(\Omega_\pm;\mathbb{R}^d)$*
$$\|\nabla M_\varepsilon^\pm(v)\|_{L^p(\Omega_\pm;\mathbb{R}^d)} \le \varepsilon^{-d/2}C_p\|v\|_{H^1(\Omega_\pm;\mathbb{R}^d)} \tag{3.11}$$

Proof. The proof of Items 1 & 2 is a direct consequence of classical results on mollifiers for $W^{1,p}(\mathbb{R}^d)$-functions, see e.g. [Bur98, p. 39, Lemma 1], combined with the continuity of the extension operator. Indeed, we have

$$\|M_\varepsilon^\pm(v)\|_{H^1(\Omega_\pm;\mathbb{R}^d)} \le \|\eta_\varepsilon * E_\pm(v)\|_{H^1(\mathbb{R}^d;\mathbb{R}^d)} \le \|\eta_1\|_{L^1(\mathbb{R}^d)}\|E_\pm(v)\|_{H^1(\mathbb{R}^d;\mathbb{R}^d)} \le C_\pm\|\eta_1\|_{L^1(\mathbb{R}^d)}\|v\|_{H^1(\Omega_\pm;\mathbb{R}^d)}\,,$$

whence (3.10) with $\overline{C} := \max\{C_+, C_-\}$, and Item 2.

Ad 3.: For the mollifiers defined in (3.8), observe that

$$\begin{aligned}
&\nabla_z\eta_1(z) = \zeta\exp(-(1-|z|^2)^{-1})(-(1-|z|^2)^{-2}2z) \quad \text{for all } z \text{ with } |z|<1,\\
&\nabla_x\eta_\varepsilon(x) = \varepsilon^{-d}\nabla_x\left(\eta_1\left(\frac{x}{\varepsilon}\right)\right) = \varepsilon^{-(d+1)}\nabla_z\eta_1\left(\frac{x}{\varepsilon}\right) \quad \text{for all } x \text{ with } |x|<\varepsilon\,.
\end{aligned} \tag{3.12}$$

Let $q' \ge 1$; using the transformation $(y-x)/\varepsilon = z$, $\mathrm{d}z_i = \varepsilon^{-1}\mathrm{d}y_i$ for $i \in \{1,\ldots,d\}$, the $L^{q'}$-norm of $\nabla\eta_\varepsilon$ reads as follows

$$\begin{aligned}
\|\nabla_x\eta_\varepsilon(x-\bullet)\|_{L^{q'}(\mathbb{R}^d)} &= \Big(\int_{\mathbb{R}^d}\big|\nabla_x\eta_\varepsilon(x-y)\big|^{q'}\,\mathrm{d}y\Big)^{1/q'}\\
&= \Big(\int_{\mathbb{R}^d}\varepsilon^{(d-q'(d+1))}\big|\nabla_z\eta_1(z)\big|^{q'}\,\mathrm{d}z\Big)^{1/q'} = \varepsilon^{(d-q'(d+1))/q'}\|\nabla_z\eta_1\|_{L^{q'}(\mathbb{R}^d)}\,.
\end{aligned} \tag{3.13}$$

For $v \in H^1(\Omega_\pm;\mathbb{R}^d)$ the above considerations are now used to estimate $\|\nabla M_\varepsilon^\pm(v)\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$. For this, we will in particular apply Hölder's inequality with the Sobolev exponent $q = 2d/(d-2)$, for which $\|v\|_{L^q(\Omega_\pm;\mathbb{R}^d)}$ is well-defined due to the continuous embedding $H^1(\Omega_\pm;\mathbb{R}^d) \subset L^q(\Omega_\pm;\mathbb{R}^d)$, i.e. there is $C_{\mathrm{S}} > 0$ such that

$$\|v\|_{L^q(\Omega_\pm;\mathbb{R}^d)} \le C_{\mathrm{S}}\|v\|_{H^1(\Omega_\pm;\mathbb{R}^d)}. \tag{3.14}$$

Furthermore, note that, for $q = 2d/(d-2)$, it is $q' = q/(q-1) = 2d/(d+2)$ and hence, $\varepsilon^{(d-q'(d+1))/q'} = \varepsilon^{-d/2}$ in (3.13) above. Thus, we obtain

$$\begin{aligned}
\|\nabla_x M_\varepsilon(v)\|^p_{L^p(\Omega_\pm;\mathbb{R}^d)} &\le \int_{\Omega_\pm}\Big(\sum_{i=1}^d\big(\int_{\mathbb{R}^d}\big|\nabla_x\eta_\varepsilon(x-y)E_\pm(v_i)(y)\big|\,\mathrm{d}y\big)^2\Big)^{p/2}\,\mathrm{d}x\\
&\le \int_{\Omega_\pm}\Big(\sum_{i=1}^d\|\nabla_x\eta_\varepsilon(x-\bullet)\|^2_{L^{q'}(\mathbb{R}^d)}\|E_\pm(v_i)\|^2_{L^q(\mathbb{R}^d)}\Big)^{p/2}\,\mathrm{d}x\\
&\le C_{d,p}\sum_{i=1}^d\int_{\Omega_\pm}\|\nabla_x\eta_\varepsilon(x-\bullet)\|^p_{L^{q'}(\mathbb{R}^d)}\|E_\pm(v_i)\|^p_{L^q(\mathbb{R}^d)}\,\mathrm{d}x\\
&\le \varepsilon^{-dp/2}C_{d,p}\|\nabla_z\eta_1\|^p_{L^{q'}(\mathbb{R}^d)}\|v\|^p_{H^1(\Omega_\pm;\mathbb{R}^d)}\,.
\end{aligned}$$

where the positive constant $C_{d,p}$, varying from the third to the fourth line, only depends on d and p, and Ω, and for the fourth estimate we have used relation (3.13), as well as the continuity of the extension and the embedding operators, cf. (3.7) and (3.14). $\square$

The recovery operator $\mathfrak{Rec}$: We now introduce the recovery operator $\mathfrak{Rec}$.

Definition 3.3 (Recovery operator $\mathfrak{Rec}$). *Suppose that M is a closed subset of $\partial\Omega_\pm$ fulfilling property $\mathfrak{a}$ from Definition 2.2. Set*

$$\begin{aligned} W^{1,r}_M(\Omega_\pm;\mathbb{R}^d) &:= \{v \in W^{1,r}(\Omega_\pm;\mathbb{R}^d),\, v = 0 \text{ on } M\},\\ d_M(x) &:= \min_{\tilde{x}\in M} |x-\tilde{x}| \quad \text{for all } x \in \overline{\Omega_\pm}. \end{aligned}$$

Let $\rho \geq 0$. Then, for all $v \in W^{1,r}_M(\Omega_\pm;\mathbb{R}^d)$ and every $x \in \overline{\Omega_\pm}$, we define

$$\mathfrak{Rec}(\rho, M, v)(x) := v(x)\xi_\rho(x) \quad \text{with} \quad \xi_\rho(x) := \min\left\{\frac{1}{\rho}(d_M(x)-\rho)^+, 1\right\}, \tag{3.15}$$

where $(\cdot)^+$ denotes the positive part, i.e. $(z)^+ := \max\{0, z\}$.

The proof that $\mathfrak{Rec}(\rho, M, v) \to v$ in $H^1(\Omega_\pm;\mathbb{R}^d)$ is based on a Hardy-type inequality recently deduced in [EHDR15, Thm. 3.4]:

Proposition 3.4 (Hardy's inequality for $r \in (1,\infty)$). *Let $\Omega_\pm$ satisfy (2.26a). Suppose that the closed set $M \subset \partial\Omega_\pm$ has Property $\mathfrak{a}$. Then, for all $r \in (1,\infty)$ there exists a constant $C_M = C(M,r)$ such that the following Hardy's inequality is fulfilled in $W^{1,r}_M(\Omega_\pm,\mathbb{R}^d)$:*

$$\forall\, v \in W^{1,r}_M(\Omega_\pm,\mathbb{R}^d): \quad \big\|v/d_M\big\|_{L^r(\Omega_\pm,\mathbb{R}^d)} \leq C_M\big\|\nabla v\big\|_{L^r(\Omega_\pm,\mathbb{R}^{d\times d})}. \tag{3.16}$$

With this Hardy's inequality at hand it is possible to deduce the following properties of $\mathfrak{Rec}$. We refer to [MRT12, Cor. 2] for the proof of Proposition 3.5 below.

Proposition 3.5 (Properties of $\mathfrak{Rec}$). *Let the assumptions of Proposition 3.4 hold true. Keep $r \in (1,\infty)$ fixed. Consider a countable family $\{\rho\}$ with $\rho \to 0$ and let $v \in W^{1,r}_M(\Omega_\pm,\mathbb{R}^d)$.*

1. *There is a constant $c_r = c_r(\Omega_\pm)$ such that for every $\rho > 0$ the following estimates hold:*

$$\|\mathfrak{Rec}(\rho,M,v)\|^r_{L^r(\Omega_\pm)} \leq \|v\|^r_{L^r(\Omega_\pm)} \quad \text{and} \quad \|\nabla\mathfrak{Rec}(\rho,M,v)\|^r_{L^r(\Omega_\pm)} \leq c_r\|\nabla v\|^r_{L^r(\Omega_\pm)}. \tag{3.17}$$

2. *$\mathfrak{Rec}(\rho,M,v) \to v$ strongly in $W^{1,r}(\Omega_\pm)$ as $\rho \to 0$.*

The bounds (3.17) will later be applied for the exponent $r = p$, whereas the strong convergence result shall be exploited for $r = 2$. As already mentioned, the recovery operator will be applied with $M = \Gamma_{\mathrm{D}}$, which is indeed required to fulfill property $\mathfrak{a}$. It will also be applied with $M = \operatorname{supp} z$, with the sequence of radii defined by (3.4). That is why, we need to impose on z the lower density estimate from Assumption 2.4 in Theorem 2.5. Assumption 2.4 is also at the basis of the following result, proved in [RT15, Prop. 6.7, 6.8], which ensures that the sequence $(\rho_k)_k$ from (3.4) tends to 0 as $k \to \infty$.

Proposition 3.6. *Assume (2.26d) on Γ_{C}. Let $(z_k)_k$, $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ fulfill (2.28) and Assumption 2.4. Then, for the sequence $(\rho_k)_k$ of radii given by (3.4) we have*

$$\operatorname{supp} z_k \subset \operatorname{supp} z + B_{\rho_k}(0) \quad \text{and} \quad \rho_k \to 0 \text{ as } k \to \infty. \tag{3.18}$$

3.2 Construction of the recovery sequence & proof of the Γ-$\limsup$ inequality

We are now in a position to carry out the construction of the recovery sequence outlined at the beginning of this Section. In order to simplify the subsequent arguments, in accordance with condition (2.26d) ensuring the "flatness" of Γ_{C}, we suppose without loss of generality that Ω is rotated in such a way that the normal n on Γ_{C} points in the x_1-direction and that the origin $0 \in \Gamma_{\mathrm{C}}$. Moreover, for every $x \in \Omega$ we may use the notation $x = (x_1, y)$ with $y = (x_2,\ldots,x_d) \in \mathbb{R}^{d-1}$. We then define the symmetric and antisymmetric parts of a function $v = (v_{\mathrm{sym}} + v_{\mathrm{anti}}) \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ via

$$v_{\mathrm{sym}}(x) := \tfrac{1}{2}\big(v(x_1,y) + v(-x_1,y)\big) \quad \text{and} \quad v_{\mathrm{anti}}(x) := \tfrac{1}{2}\big(v(x_1,y) - v(-x_1,y)\big). \tag{3.19}$$

In particular, $v_{\rm sym} \in H^1(\Omega, \mathbb{R}^d)$. Moreover, for $v \in H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ with $\Phi_\infty(v,z) < \infty$, there holds $v_{\rm anti} = 0$ a.e. on $\operatorname{supp} z$.

With our next result we give the precise definition of the recovery sequence and prove the Γ-lim sup inequality for the functionals Φ_k and $\Phi_k^{\rm adh}$.

Theorem 3.7. *Let Assumptions* (2.26) *be satisfied. Let* $(z_k)_k, z \in \mathrm{SBV}(\Gamma_{\rm C};\{0,1\})$ *satisfy* (2.28) *and Assumption 2.4. Let* $(\rho_k)_k$ *be defined by* (3.4). *For every* $k \in \mathbb{N}$ *set* $r_k := \frac{\gamma}{4k}$, *with* $\gamma = \operatorname{dist}(\Gamma_{\rm D}, \Gamma_{\rm C})$, *and consider* M_{ε_k} *from* (3.9) *with* $\varepsilon_k := k^{-\alpha}$ *for* $\alpha \in (0, 2/d)$. *Then, for* $v \in H^1_{\rm D}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ *with* $\Phi_\infty(v,z) < \infty$, *set*

$$v_k := \tilde{v}_k^{\rm sym} + \mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v}_k^{\rm anti})\,, \tag{3.20}$$

with $\tilde{v}_k$ *from* (3.5), $(\rho_k)_k$ *from* (3.4), *and the recovery operator* $\mathfrak{Rec}$ *from* (3.15). *Then, for the functionals from* (2.21)–(2.23) *there holds*

$$\lim_{k\to\infty} \Phi_k(v_k, z_k) = \Phi_\infty(v,z) \quad \textit{and} \quad \lim_{k\to\infty} \Phi_k^{\rm adh}(v_k, z_k) = \Phi_\infty(v,z)\,. \tag{3.21}$$

Proof. First of all, recall that both $M_\varepsilon^\pm$ from (3.9) and $\mathfrak{Rec}(\rho, M, \cdot)$ from (3.15) are linear operators. Hence, in (3.20) we have

$$v_k^\pm = M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, v_{\rm sym}|_{\Omega_\pm})) + \mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, v_{\rm anti}|_{\Omega_\pm})))\,. \tag{3.22}$$

With $\tilde{v} \in H^1(\Omega_\pm, \mathbb{R}^d)$ as a placeholder for $u_{\rm sym}|_{\Omega_\pm}$, resp. $u_{\rm anti}|_{\Omega_\pm}$, and using (3.10), we deduce

$$\begin{aligned}
&\|M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v})) - \tilde{v}\|_{H^1(\Omega_\pm)}\\
&\le \|M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v})) - M_{\varepsilon_k}^\pm(\tilde{v})\|_{H^1(\Omega_\pm)} + \|M_{\varepsilon_k}^\pm(\tilde{v}) - \tilde{v}\|_{H^1(\Omega_\pm)}\\
&\le \overline{C}\|\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}) - \tilde{v}\|_{H^1(\Omega_\pm)} + \|M_{\varepsilon_k}^\pm(\tilde{v}) - \tilde{v}\|_{H^1(\Omega_\pm)} \to 0\,,
\end{aligned} \tag{3.23}$$

and both terms on the right-hand side tend to 0 according to Propositions 3.2 & 3.5, since both sequences $(\varepsilon_k)_k$ and $(r_k)_k$ are null and since $r_k = \gamma/(4k) < \operatorname{dist}(\Gamma_{\rm D}, \Gamma_{\rm C})$ by assumption. Furthermore, thanks to (3.11), the L^p-norm of the gradient can be estimated as follows

$$\|\nabla M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))\|_{L^p(\Omega_\pm)} \le \varepsilon_k^{-d/2} C_{d,p} \|\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v})\|_{H^1(\Omega_\pm)} \le \varepsilon_k^{-d/2} C\,. \tag{3.24}$$

Estimate (3.23) implies that

$$M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, v_{\rm sym}|_{\Omega_\pm})) \to v_{\rm sym}|_{\Omega_\pm} \quad \text{strongly in } H^1(\Omega_\pm, \mathbb{R}^d). \tag{3.25}$$

Moreover, by estimate (3.24) we conclude that

$$k^{-p}\|\nabla M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))\|_{L^p(\Omega_\pm)} \le k^{-p} \varepsilon_k^{-dp/2} C^p \to 0 \quad \text{as } k \to \infty, \tag{3.26}$$

due to $\varepsilon_k = k^{-\alpha}$ with $\alpha \in (0, 2/d)$.

It remains to verify similar relations for the term involving $v_{\rm anti}|_{\Omega_\pm}$, again abbreviated with $\tilde{v}$. With the aid of (3.17) and the linearity of $\mathfrak{Rec}$, we obtain

$$\begin{aligned}
&\|\mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))) - \tilde{v}\|_{H^1(\Omega_\pm)}\\
&\le \|\mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))) - \mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v})\|_{H^1(\Omega_\pm)} + \|\mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v}) - \tilde{v}\|_{H^1(\Omega_\pm)}\\
&\le C\|M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v})) - \tilde{v}\|_{H^1(\Omega_\pm)} + \|\mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v}) - \tilde{v}\|_{H^1(\Omega_\pm)} \to 0
\end{aligned} \tag{3.27}$$

by (3.23) and Proposition 3.5. In order to deduce an estimate for the L^p-norm of the gradient we rewrite $\mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))) = \xi_{\rho_k}^{\operatorname{supp} z} M_{\varepsilon_k}^\pm(\xi_{r_k}^{\Gamma_{\rm D}^\pm} v)$ with the aid of (3.15), and hence find that $\nabla\mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v}))) = \xi_{\rho_k}^{\operatorname{supp} z}\nabla M_{\varepsilon_k}^\pm(\xi_{r_k}^{\Gamma_{\rm D}^\pm}\tilde{v}) + M_{\varepsilon_k}^\pm(\xi_{r_k}^{\Gamma_{\rm D}^\pm}\tilde{v}) \otimes \nabla\xi_{\rho_k}^{\operatorname{supp} z}$. Thus, by (3.17) and (3.11) it is

$$\begin{aligned}
\|\nabla\mathfrak{Rec}(\rho_k, \operatorname{supp} z, M_{\varepsilon_k}^\pm(\mathfrak{Rec}(r_k, \Gamma_{\rm D}^\pm, \tilde{v})))\|_{L^p(\Omega_\pm)} &\le \|\nabla M_{\varepsilon_k}^\pm(\xi_{r_k}^{\Gamma_{\rm D}^\pm}\tilde{v})\|_{L^p(\Omega_\pm)} + \|M_{\varepsilon_k}^\pm(\xi_{r_k}^{\Gamma_{\rm D}^\pm}\tilde{v}) \otimes \nabla\xi_{\rho_k}^{\operatorname{supp} z}\|_{L^p(\Omega_\pm)}\\
&\le \varepsilon_k^{-d/2}(C_{d,p} + C)\|\xi_{r_k}^{\Gamma_{\rm D}^\pm}\tilde{v}\|_{H^1(\Omega_\pm)} \le \varepsilon_k^{-d/2} C'\,.
\end{aligned} \tag{3.28}$$

Let us now conclude the proof of (3.21). It follows from (3.23) and (3.27) that $v_k \to v$ strongly in $H^1(\Omega\backslash\Gamma_C, \mathbb{R}^d)$. Hence we can choose a (not relabeled) subsequence that converges pointwise a.e. in $\Omega\backslash\Gamma_C$. Then, for the quadratic part W_2 of the elastic energy we easily conclude that

$$\int_{\Omega\backslash\Gamma_C} W_2(e(v_k))\,\mathrm{d}x \to \int_{\Omega\backslash\Gamma_C} W_2(e(v))\,\mathrm{d}x \tag{3.29}$$

via the the dominated convergence theorem. As for the term $k^{-p}W_p$, we have that

$$\int_{\Omega\backslash\Gamma_C} k^{-p}W_p(e(v_k))\,\mathrm{d}x \to 0\,. \tag{3.30}$$

due to growth property of W_p in combination with estimates (3.24) & (3.28). Finally, there holds

$$z_k [\![v_k]\!] = z_k [\![\tilde{v}_k^{\mathrm{sym}}]\!] + z_k [\![\mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v}_k^{\mathrm{anti}})]\!] = 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_C, \tag{3.31}$$

since for the symmetric part we have $[\![\tilde{v}_k^{\mathrm{sym}}]\!] = 0$ a.e. on Γ_C, while, by construction, $[\![\mathfrak{Rec}(\rho_k, \operatorname{supp} z, \tilde{v}_k^{\mathrm{anti}})]\!] = 0$ on $\operatorname{supp} z + B_{\rho_k}$ which contains $\operatorname{supp} z_k$, cf. (3.18). Since the functions z_k fulfill the lower density estimate from Assumption 2.4, Lemma 3.1 is applicable. Therefore, from (3.31) we infer that $[\![v_k]\!] = 0$ a.e. on $\operatorname{supp} z_k$, i.e. that

$$\text{both } \mathcal{J}_\infty([\![v_k]\!], z_k) = 0 \quad \text{and } \mathcal{J}_k([\![v_k]\!], z_k) = 0 \quad \text{for every } k \in \mathbb{N}. \tag{3.32}$$

From (3.29), (3.30), and (3.32) we conclude (3.21) and thus complete the proof. □

4 Applications

4.1 From nonlinear to linear elasticity in the brittle delamination system

Let us now address the limit passage from nonlinear to linear (small-strain) elasticity in the coupled rate-dependent/independent system for brittle delamination consisting of

1. the mechanical force balance for the displacements (2.3), with the stored elastic energy density $W(e) = W_2(e) + \frac{1}{k^p}W_p(e)$, where we let $k \to \infty$;
2. the contact boundary condition (2.4);
3. the brittle delamination flow rule (2.7).

Due to the rate-independent character of the flow rule, which possibly leads to jump discontinuities of z as a function of time, system (2.3, 2.4, 2.7) has to be weakly formulated. As already mentioned in Sec. 2, for this we resort to the notion of *semistable energetic solution* for coupled rate-dependent/independent systems, first proposed in [Rou09] for rate-independent processes in viscous solids, and recently extended and generalized in [RT17a]. We now recall this definition in the context of

- the *nonlinearly elastic* brittle delamination system, i.e. (2.3, 2.4, 2.7) with $W(e) = W_2(e)+\frac{1}{k^p}W_p(e)$;
- the *linearly elastic* brittle delamination system, i.e. (2.3, 2.4, 2.7) with $W(e) = W_2(e)$,

where, of course, the terms 'nonlinearly elastic' and 'linearly elastic' have been used with slight abuse, only to refer to the nonlinear/linear character of the equation for the displacements (at small strains).

Prior to giving Definition 4.1, we need to fix our conditions on the forces F and f: we assume that $F \in W^{1,1}(0,T; H^1_D(\Omega\backslash\Gamma_C;\mathbb{R}^d)^*)$ and $f \in W^{1,1}(0,T; L^{2(d-1)/d}(\Gamma_N;\mathbb{R}^d))$, so that the total loading L defined by (2.9) fulfills

$$L \in W^{1,1}(0,T; H^1_D(\Omega\backslash\Gamma_C;\mathbb{R}^d)^*)\,. \tag{4.1}$$

We then introduce the energy functionals driving the nonlinearly and linearly elastic systems, respectively:

$$\begin{aligned}
&\mathcal{E}_k,\ \mathcal{E}_\infty : [0,T] \times H^1_D(\Omega\backslash\Gamma_C;\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_C;\{0,1\}), \to (-\infty,\infty] \\
&\quad \mathcal{E}_k(t,u,z) := \Phi_k(u,z) + \mathcal{G}(z) - \langle L(t), z\rangle_{H^1_D(\Omega\backslash\Gamma_C;\mathbb{R}^d)}, \\
&\quad \mathcal{E}_\infty(t,u,z) := \Phi_\infty(u,z) + \mathcal{G}(z) - \langle L(t), z\rangle_{H^1_D(\Omega\backslash\Gamma_C;\mathbb{R}^d)},
\end{aligned} \tag{4.2}$$

with $\mathcal{G}$ defined by (2.16). Finally, we consider the dissipation potential

$$\mathcal{R} : L^1(\Gamma_{\mathrm{C}}) \to [0,\infty], \qquad \mathcal{R}(\dot{z}) := \int_{\Gamma_{\mathrm{C}}} \mathrm{R}(\dot{z})\,\mathrm{d}x\,, \quad \text{with } \mathrm{R}(v) := \begin{cases} a_1|v| & \text{if } v \leq 0, \\ \infty & \text{otherwise} \end{cases} \tag{4.3}$$

and $a_1 > 0$. The fact that $\mathrm{R}(v) = \infty$ if $v > 0$ ensures the unidirectionality of the delamination process, i.e. a crack can only increase or stagnate but its healing is excluded. With $\mathcal{R}$ we associate the total variation functional

$$\mathrm{Var}_{\mathcal{R}}(z;[s,t]) := \sup\left\{ \sum_{j=1}^{N} \mathcal{R}(z(r_j){-}z(r_{j-1})) \ : \quad s = r_0 < r_1 < \ldots < r_{N-1} < r_N = t \right\}.$$

for all $[s,t] \subset [0,T]$. Observe that the unidirectionality encoded in $\mathcal{R}$ provides monotonicity with respect to time of functions z with $\mathrm{Var}_{\mathcal{R}}(z;[s,t]) < \infty$. Hence, $\mathrm{Var}_{\mathcal{R}}(z;[s,t]) = \mathcal{R}(z(t) - z(s))$ in this case.

We are now in a position to give the following

Definition 4.1. *We say that a pair (u,z), with $u : [0,T] \to W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ in the nonlinear case and $u : [0,T] \to H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ for the linear case, and $z : [0,T] \to \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$, is a* semistable energetic *solution of the nonlinearly/linearly elastic brittle delamination system, if*

$$u \in H^1(0,T;H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) \cap \begin{cases} L^\infty(0,T;W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) & \textit{in the nonlinear case,} \\ L^\infty(0,T;H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) & \textit{in the linear case,} \end{cases}$$
$$z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})) \cap \mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}})),$$

the pair (u,z) fulfills

- *the weak formulation (2.8) of the mechanical force balance, with $q = p > d$ for the nonlinear case and $q = 2$ for the linear one;*
- *the semistability condition*

$$\mathcal{E}_k(t,u(t),z(t)) \leq \mathcal{E}_k(t,u(t),\tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) \quad \textit{for all } \tilde{z} \in L^1(\Gamma_{\mathrm{C}}) \textit{ and all } t \in [0,T], \tag{4.4}$$

- *the energy-dissipation inequality for all $t \in [0,T]$*

$$\mathrm{Var}_{\mathcal{R}}(z;[0,t]) + \int_0^t \mathbb{D}e(\dot{u}) : e(\dot{u})\,\mathrm{d}x + \mathcal{E}_k(t,u(t),z(t)) \leq \mathcal{E}_k(0,u(0),z(0)) + \int_0^t \partial_t\mathcal{E}_k(r,u(r),z(r))\,\mathrm{d}r, \tag{4.5}$$

with $k \in \mathbb{N}$ ($k = \infty$) for the nonlinearly (linearly, respectively) elastic system.

Note that the existence of semistable energetic solutions to the nonlinearly elastic brittle system was proved in [RT15].

The following result formalizes the limit passage from nonlinear to linear elasticity for semistable energetic solutions of the brittle delamination system. For technical reasons that will be expounded in the proof, we need to strengthen our Assumption 2.3 on the domain, by requiring in addition that Γ_{C} is *convex.*

Theorem 4.2. *Under Assumption 2.3 suppose, in addition, that Γ_{C} is convex. Let $(u_0^k, z_0^k)_k \subset W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)\times \mathrm{SBV}(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ be a sequence of data for the nonlinearly elastic brittle systems, and suppose that*

$$(u_0^k, z_0^k) \to (u_0, z_0) \quad \textit{in } H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_{\mathrm{C}};\mathbb{R}^d) \textit{ with } \mathcal{E}_k(u_0^k, z_0^k) \to \mathcal{E}_\infty(u_0,z_0) \quad \textit{as } k \to \infty. \tag{4.6a}$$

Also, suppose that (u_0, z_0) fulfill the semistability condition at $t = 0$

$$\mathcal{E}(0,u_0,z_0) \leq \mathcal{E}(0,u_0,\tilde{z}) + \mathcal{R}(\tilde{z}{-}z_0) \quad \textit{for all } \tilde{z} \in L^1(\Gamma_{\mathrm{C}}). \tag{4.6b}$$

Let $(u_k, z_k)_k$ be a sequence of semistable energetic solutions of the nonlinearly elastic brittle system emanating from the initial data $(u_0^k, z_0^k)_k$. Then, there exist a (not relabeled) subsequence and functions $u \in H^1(0,T; H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$ and $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})) \cap \mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}}))$ such that

$$
\begin{aligned}
&u_k \rightharpoonup u && \text{in } H^1(0,T;H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)),\\
&u_k(t) \rightharpoonup u(t) && \text{in } H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \quad \text{for all } t\in[0,T],\\
&z_k \overset{*}{\rightharpoonup} z && \text{in } L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))\cap L^\infty((0,T)\times\Gamma_{\mathrm{C}}),\\
&z_k(t) \overset{*}{\rightharpoonup} z(t) && \text{in } \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})\cap L^\infty(\Gamma_{\mathrm{C}}) \quad \text{for all } t\in[0,T],
\end{aligned}
\tag{4.7}
$$

$u(0) = u_0$, $z(0) = z_0$, and the pair (u,z) is a semistable energetic solution of the linearly elastic brittle system in the sense of Def. 4.1.

Remark 4.3 (Alternative scaling & energy-dissipation balance)**.** In [RTP15, RT15, RT17b] also an alternative scaling for certain energy contributions was investigated. More, precisely, we replaced the perimeter regularization $\mathcal{G}$ in (4.2) and dissipation potential $\mathcal{R}$ in (4.3) by their scaled versions

$$
\mathcal{G}_k(z) := \tfrac{1}{k}\mathcal{G}(z) \quad \text{and} \quad \mathcal{R}_k(v) := \tfrac{1}{k}\mathcal{R}(v)\,. \tag{4.8}
$$

In [RTP15] this was shown to be beneficial for modeling the onset of rupture when performing the adhesive contact approximation of brittle delamination. Still, the associated semistability inequality yielded compactness for the perimeters and the dissipation terms of the approximate solutions, as can be verified by a multiplication with a factor k. The uniform bound on the perimeters independent of k thus entailed that $\mathcal{G}_k(z_k(t)) \to 0$ along semistable energetic solutions as $k\to\infty$. Thus, given that the initial data are well-prepared, it was possible in [RTP15] to deduce an energy-dissipation balance for the limit system. A similar result is also expected if the scaling (4.8) is applied in the setup presented in Theorem 4.2.

Sketch of the proof of Theorem 4.2. We will not develop the proof in its completeness but rather highlight its main ingredients, focusing in particular on the limit passage in the mechanical force balance for the displacements. We will often refer to [RT15] for all details. We now split the proof into five steps.

Step 0: A priori estimates and compactness: Exploiting regularity assumption (4.1), which allows us to estimate the work of the external loadings, as well as the information that $\sup_{k\in\mathbb{N}} \mathcal{E}_k(u_0^k, z_0^k) \le C < \infty$, from the energy-dissipation inequality for the nonlinearly elastic case (i.e. $k\in\mathbb{N}$), written on the interval $[0,T]$, we deduce that

$$
\exists\, C>0\ \forall\, k\in\mathbb{N}: \qquad \mathrm{Var}_{\mathcal{R}}(z_k;[0,t]) + \int_0^t \mathbb{D}e(\dot u_k):e(\dot u_k)\,\mathrm{d}x + \sup_{t\in[0,T]} |\mathcal{E}_k(t,u_k(t),z_k(t))| \le C\,. \tag{4.9}
$$

This yields the uniform bounds

$$
\sup_{k\in\mathbb{N}} \Big(\|u_k\|_{H^1(0,T;H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))} + \|z_k\|_{L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))\cap \mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}}))} \Big) \le C,
$$

also by exploiting Korn's inequality for the displacements. Then, standard compactness arguments imply convergences (4.7), cf. the proof of [RT15, Thm. 4.3], which in particular give $u(0) = u_0$, $z(0) = z_0$. It also follows from (4.7), via standard lower semicontinuity arguments, that

$$
\liminf_{k\to\infty} \mathcal{E}_k(t,u_k(t),z_k(t)) \ge \mathcal{E}_\infty(t,u(t),z(t)) \quad \text{for every } t\in[0,T]. \tag{4.10}
$$

Step 1: Fine properties of the semistable sequence $(z_k)_k$. Exploiting the additional condition that Γ_{C} is convex, in [RT15, Thm. 6.6] it was proved that the semistability condition (4.4) guarantees the validity of the lower density estimate (2.27) for every $k\in\mathbb{N}\cup\{\infty\}$, with constants uniform w.r.t. $k\in\mathbb{N}\cup\{\infty\}$. Therefore, the sequence $(z_k)_k$ fulfills Assumption 2.4 of Theorem 2.5.

Step 2: Limit passage in the mechanical force balance for the displacements. We apply Theorem 2.5 and conclude the MOSCO-convergence of the functionals $\Phi_k(\cdot, z_k)$ to $\Phi(\cdot, z)$ w.r.t. the topology of $H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$. Then, in order to pass to the limit in the mechanical force balance (2.8) as $k \to \infty$, we easily adapt the arguments from the proof of [RT15, Prop. 5.6]. They are based on the fact that, for $k \in \mathbb{N}$, the weak formulation (2.8) can be reformulated in terms of the subdifferential (in the sense of convex analysis) of Φ_k w.r.t. the variable u, namely $\partial_u \Phi_k : H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) \times \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \rightrightarrows H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))^*$ given by

$$\xi \in \partial_u \Phi_k(u,z) \text{ if and only if } u \in W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \text{ and}$$
$$\langle \xi, v\rangle_{W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)} = \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathrm{D}W_2(e(u)) + k^{-p}\mathrm{D}W_p(e(u))\big) : e(v)\,\mathrm{d}x + \langle \lambda, v\rangle_{H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)}$$

for all $v \in W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$, with λ an element of the subdifferential $\partial_u(\mathcal{I}_C + \mathcal{J}_k(\cdot, z)) : H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \rightrightarrows H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*$. Then, the nonlinearly elastic version of the mechanical force balance (2.8) is equivalent to

$$\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathbb{D}\dot{e}(t) + \mathrm{D}W_2(e(u)) + k^{-p}\mathrm{D}W_p(e(u))\big) : e(v)\,\mathrm{d}x + \langle \lambda(t), v\rangle_{H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)} = \langle L(t), v\rangle_{H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)} \tag{4.11}$$

for all $v \in W^{1,p}_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$, with $\lambda(t)$ a selection in $\partial_u(\mathcal{I}_C + \mathcal{J}_k(\cdot, z(t)))(u(t))$. Analogously, in the linearly elastic case (2.8) reformulates in terms of the subdifferential $\partial_u \Phi_\infty : H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \times \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}) \rightrightarrows H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*$. Now, the MOSCO-convergence of the functionals $\Phi_k(\cdot, z_k)$ to $\Phi_\infty(\cdot, z)$ guarantees the *convergence in the sense of graphs* of the corresponding subdifferentials $\partial_u\Phi_k(\cdot, z_k)$ to $\partial_u\Phi_\infty(\cdot, z)$, cf. [Att84]. This is the key observation for passing to the limit in (4.11), arguing in the very same way as for [RT15, Prop. 5.6]. These arguments also yield, as a by-product, that

$$u_k(t) \to u(t) \text{ in } H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \text{ and } k^{-p}\int_{\Omega\backslash\Gamma_{\mathrm{C}}} W_p(e(u_k(t))\,\mathrm{d}x \to 0 \text{ as } k \to \infty \quad \text{for a.a. } t \in (0,T),$$

hence

$$\Phi_k(u_k(t), z_k(t)) \to \Phi_\infty(u(t), z(t)) \qquad \text{as } k \to \infty \quad \text{for a.a. } t \in (0,T). \tag{4.12}$$

Step 3: Limit passage in the semistability condition. First of all, observe that, for $k \in \mathbb{N} \cup \{\infty\}$ condition (4.4) reduces to

$$\mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, z_k(t)) + \mathcal{G}(z_k(t)) \le \mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, \tilde{z}) + \mathcal{G}(\tilde{z}) + \mathcal{R}(\tilde{z} - z_k(t)) \quad \text{for all } \tilde{z} \in L^1(\Gamma_{\mathrm{C}}) \tag{4.13}$$

for all $t \in [0,T]$. We now aim to pass to the limit as $k \to \infty$ in (4.13) for every $t \in (0,T]$ (the semistability condition holds at $t = 0$ thanks to (4.6b)) and show that the functions (u,z) fulfill it for $k = \infty$. Following a well consolidated procedure for energetic solutions to purely rate-independent systems (cf. [MRS08]), for $t \in (0,T]$ fixed and given $\tilde{z} \in L^1(\Gamma_{\mathrm{C}})$ such that $\mathcal{R}(\tilde{z} - z(t)) < \infty$ and $\mathcal{J}_\infty(\llbracket u(t)\rrbracket, \tilde{z}) + \mathcal{G}(\tilde{z}) < \infty$ (otherwise (4.13) trivially holds), we exhibit a recovery sequence $(\tilde{z}_k)_k$, suitably converging to $\tilde{z}$ and fulfilling

$$\begin{aligned}
&\limsup_{k\to\infty} \big(\mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, \tilde{z}_k) + \mathcal{G}(\tilde{z}_k) + \mathcal{R}(\tilde{z}_k - z_k(t)) - \mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, z_k(t)) - \mathcal{G}(z_k(t))\big) \\
&\le \mathcal{J}_\infty(\llbracket u(t)\rrbracket, \tilde{z}) + \mathcal{G}(\tilde{z}) + \mathcal{R}(\tilde{z} - z(t)) - \mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, z) - \mathcal{G}(z(t))\,.
\end{aligned} \tag{4.14}$$

For this, we borrow the construction from the proof of [RT15, Prop. 5.9] and set

$$\tilde{z}_k := \tilde{z}\chi_{A_k} + z_k(1 - \chi_{A_k}) \quad \text{with } A_k := \{x \in \Gamma_{\mathrm{C}} \,:\, 0 \le \tilde{z}(x) \le z_k(x)\}$$

and χ_{A_k} its characteristic function. Observe that $0 \le \tilde{z}_k \le z_k$ a.e. on Γ_{C} by construction, therefore from $\sup_k \sup_{t\in(0,T)} \mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, z_k(t)) = 0$ due to (4.9) we gather that $\mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, \tilde{z}_k) = 0$ for all $k \in \mathbb{N}$. Therefore,

$$\limsup_{k\to\infty} \big(\mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, \tilde{z}_k) - \mathcal{J}_\infty(\llbracket u_k(t)\rrbracket, z_k(t))\big) = 0 = \mathcal{J}_\infty(\llbracket u(t)\rrbracket, \tilde{z}) - \mathcal{J}_\infty(\llbracket u(t)\rrbracket, z(t))\,.$$

We refer to the proof of [RT15, Prop. 5.9] for the calculations on the remaining contributions to (4.14).

Step 4: Proof of the energy-dissipation inequality (4.5)**.** It follows by taking the $\liminf_{k\to\infty}$ of (4.5) for the nonlinearly elastic brittle system. For the left-hand side, we rely on convergences (4.7), the lower semicontinuity properties of the dissipative contributions to (4.5), and (4.10). For the right-hand side, we resort to the energy convergence (4.6a) for the initial data and to the continuity properties of the power term $\partial_t\mathcal{E}$, in view of (4.1).

This concludes the proof of Theorem 4.2. $\square$

4.2 The joint discrete-to-continuous and adhesive-to-brittle limit in the mechanical force balance of the thermoviscoelastic system

In this final section we shortly discuss how the Mosco-convergence statement of Theorem 2.5 concerning the functionals $(\Phi_k^{\mathrm{adh}})_k$ from (2.22) can be used to prove the existence of solutions for a model for brittle delamination, also encompassing thermal effects. More precisely, the evolution of the displacement u, of the delamination variable z, and of the absolute temperature ϑ is governed by the following PDE system:

$$
\begin{aligned}
&-\operatorname{div}\sigma(e,\dot e,\vartheta) = F && \text{in } (0,T)\times(\Omega_+\cup\Omega_-), && (4.15a)\\
&\dot\vartheta - \operatorname{div}\big(\mathbb{K}(e,\vartheta)\nabla\vartheta\big) = \dot e{:}\mathbb{D}{:}\dot e - \vartheta\mathbb{B}{:}\dot e + G && \text{in } (0,T)\times(\Omega_+\cup\Omega_-), && (4.15b)\\
&u = 0 && \text{on } (0,T)\times\Gamma_{\mathrm{D}}, && (4.15c)\\
&\sigma(e,\dot e,\vartheta)\big|_{\Gamma_{\mathrm{N}}}\mathbf{n} = f && \text{on } (0,T)\times\Gamma_{\mathrm{N}}, && (4.15d)\\
&(\mathbb{K}(e,\theta)\nabla\theta)\mathbf{n} = g && \text{on } (0,T)\times\partial\Omega, && (4.15e)\\
&\sigma(e,\dot e,\vartheta)|_{\Gamma_{\mathrm{C}}}\mathbf{n} + \partial_u\widetilde{J}_\infty([\![u]\!],z) + \partial I_{C(x)}([\![u]\!]) \ni 0 && \text{on } (0,T)\times\Gamma_{\mathrm{C}}, && (4.15f)\\
&\partial\mathrm{R}(\dot z) + \partial\mathcal{G}(z) + \partial_z\widetilde{J}_\infty([\![u]\!],z) \ni 0 && \text{on } (0,T)\times\Gamma_{\mathrm{C}}, && (4.15g)\\
&\tfrac12\big(\mathbb{K}(e,\vartheta)\nabla\vartheta|_{\Gamma_{\mathrm{C}}}^+ + \mathbb{K}(e,\vartheta)\nabla\vartheta|_{\Gamma_{\mathrm{C}}}^-\big)\cdot\mathbf{n} + \eta([\![u]\!],z)[\![\vartheta]\!] = 0 && \text{on } (0,T)\times\Gamma_{\mathrm{C}}, && (4.15h)\\
&[\![\mathbb{K}(e,\vartheta)\nabla\vartheta]\!]\cdot\mathbf{n} = -a_1\dot z && \text{on } (0,T)\times\Gamma_{\mathrm{C}}. && (4.15i)
\end{aligned}
$$

Here, the stress tensor σ encompasses both Kelvin-Voigt rheology and thermal expansion in a *linearly elastic* way, i.e.

$$\sigma(e,\dot e,\vartheta) = \mathbb{D}\dot e + \mathrm{D}W_2(e) - \theta\mathbb{B}. \tag{4.16}$$

The heat equation (4.15b), featuring the positive definite matrix of heat conduction coefficients $\mathbb{K}(e,\vartheta)$ and the positive heat source G, is complemented by the two boundary conditions (4.15h) and (4.15i) (with $g\geq 0$ another external heat source on the boundary $\partial\Omega$), which balance the heat transfer across Γ_{C} with the ongoing crack growth. In particular, the function η is a heat-transfer coefficient, determining the heat convection through Γ_{C}, which depends on the state of the bonding and on the distance between the crack lips.

In [RT15] we proved the existence of semistable energetic solutions (with the heat equation formulated in a suitably weak way) for system (4.15) in the *nonlinearly elastic* (small-strain) case, i.e. with $\sigma(e,\dot e,\vartheta) = \mathbb{D}\dot e + \mathrm{D}W_p(e) - \theta\mathbb{B}$ and $p > d$. As explained in Section 2, the latter constraint can be now overcome. Nonetheless, in order to show the existence of solutions to system (4.15) with (4.16), it is necessary to resort to a *nonlinear* approximation of the mechanical force for the displacements.

In fact, mimicking [RR11, RT15] one can construct approximate solutions for system (4.15) with (4.16) by a carefully devised time discretization scheme, illustrated below (however neglecting the boundary conditions). In this scheme the equation for the displacements is discretized in the following way

$$-\operatorname{div}\left(\mathbb{D}e\left(\frac{u_\tau^j - u_\tau^{j-1}}{\tau}\right) + \mathrm{D}W_2(e(u_\tau^j)) + \tau\mathrm{D}W_p(e(u_\tau^j)) - \vartheta_\tau^j\mathbb{B}\right) = F_\tau^j \quad \text{in } \Omega_+\cup\Omega_-, \tag{4.17a}$$

where τ is the time-step associated with a (for simplicity equidistant) partition $\{0 = t_\tau^0 < t_\tau^1 < \ldots < t_\tau^j < \ldots < t_\tau^{J_\tau} = T\}$ of the interval $[0,T]$ and $F_\tau^j = \frac{1}{\tau}\int_{t_\tau^{j-1}}^{t_\tau^j} F(s)\,\mathrm{d}s$. The nonlinear regularizing term $\mathrm{D}W_p(e(u_\tau^j))$, with $p > 4$, is added to the discrete momentum balance in order to compensate the quadratic growth of the terms on the right-hand side of the (discretized) heat equation, namely

$$\frac{\vartheta_\tau^j - \vartheta_\tau^{j-1}}{\tau} - \mathrm{div}\left(\mathbb{K}(e(u_\tau^j), \vartheta_\tau^j)\nabla\vartheta_\tau^j\right) = e\left(\frac{u_\tau^j - u_\tau^{j-1}}{\tau}\right):\mathbb{D}:e\left(\frac{u_\tau^j - u_\tau^{j-1}}{\tau}\right) - \vartheta_\tau^j\mathbb{B}:e\left(\frac{u_\tau^j - u_\tau^{j-1}}{\tau}\right) + G_\tau^j \tag{4.17b}$$

in $\Omega_+\cup\Omega_-$, with G_τ^j defined by local means like F_τ^j. In this way, the right-hand side of (4.17b) turns out to be in $L^2(\Omega)$, and classical Leray-Schauder fixed point arguments can be applied to prove the existence of solutions to (4.17a,4.17b). Finally, we mention that the flow rule for the delamination parameter is discretized and further approximated by penalizing the brittle constraint, i.e. replacing $\widetilde{J}_\infty$ in (4.15g) by J_k.

Semistable energetic solutions of the time-continuous system (4.15), with (4.16), then arise from taking the limit of its time-discrete version, as $\tau \downarrow 0$ and $k \to \infty$ *simultaneously*. Without entering into the analysis of the heat equation and of the delamination flow rule, let us only comment on the limit passage in the weak formulation of the (discrete) equation for the displacements. For that, a key role is played the Mosco-convergence properties as $k \to \infty$ of the functionals

$$\Phi_k^{\mathrm{adh}}(u,z) := \begin{cases} \int_{\Omega\setminus\Gamma_\mathrm{C}} \left(W_2(e(u)) + \tau_k W_p(e(u))\right)\,\mathrm{d}x + \mathcal{J}_k([\![u]\!], z) & \text{if } u \in W_\mathrm{D}^{1,p}(\Omega\setminus\Gamma_\mathrm{C};\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

with $(\tau_k = k^{-p})_k$ a null sequence as $k \to \infty$. We have denoted the above functionals with the same symbol used for the functionals (2.22), to highlight that Theorem 2.5 holds for them as well and guarantees the Mosco-convergence of the functionals $(\Phi_k^{\mathrm{adh}})_k$ to Φ_∞ from (2.23), and thus the limit passage in the mechanical force balance for the displacements.

Acknowledgements. This work has been partially supported by the DFG within the project *Finite element approximation of functions of bounded variation and application to models of damage, fracture, and plasticity* in the Priority Programme SPP 1748 *Reliable Simulation Techniques in Solid Mechanics. Development of Non-standard Discretisation Methods, Mechanical and Mathematical Analysis.* During a research stay of M.T. at DICATAM, University of Brescia, and of R.R. at WIAS Berlin major steps were made for the progress of this work. The two authors acknowledge the kind hospitality at their host institutions.

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